

Combining convex-concave decompositions and linearization approaches for solving BMIs, with application to static output feedback

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Abstract—A novel optimization method is proposed to minimize a convex function subject to bilinear matrix inequality (BMI) constraints. The key idea is to decompose the bilinear mapping as a difference between two positive semidefinite convex mappings. At each iteration of the algorithm the concave part is linearized, leading to a convex subproblem. Applications to various output feedback controller synthesis problems are presented. In these applications the subproblem in each iteration step can be turned into a convex optimization problem with linear matrix inequality (LMI) constraints. The performance of the algorithm has been benchmarked on the data from the COMPlib library.

Index Terms—Static feedback controller design, linear time-invariant system, bilinear matrix inequality, semidefinite programming, convex-concave decomposition.

I. INTRODUCTION

OPTIMIZATION involving matrix constraints has a broad interest and applications in static state/output feedback controller design, robust stability of systems, topology optimization, see, e.g. [3], [5], [16], [18]. Many problems in these fields can be reformulated as an optimization problem with linear matrix inequality (LMI) constraints [5], [18] which can be solved efficiently and reliably by means of interior point methods for semidefinite programming (SDP) [3], [21] and efficient open-source software tools such as Sedumi [27] and SDPT3 [29]. However, solving optimization problems involving nonlinear matrix inequality constraints is still a big challenge in practice. The methods and algorithms for nonlinear matrix constrained optimization problems are still limited [8], [10], [16].

In control theory, many problems related to the design of a reduced-order controller can be conveniently reformulated as a feasibility problem or an optimization problem with bilinear matrix inequality (BMI) constraints by means of, for instance, Lyapunov's theory. The BMI constraints make the problems much more difficult than the LMI ones due to their

nonconvexity and possible nonsmoothness. It has been shown in [4] that the optimization problems involving BMI are NP-hard. Several approaches to solve optimization problems with BMI constraints have been proposed. For instance, Goh *et al* [11] considered problems in robust control by means of BMI optimization using global optimization methods. Hol *et al* in [15] proposed to use a sum-of-squares approach to fixed order H -infinity synthesis. Apkarian and Tuan [2] proposed local and global methods for solving BMIs also based on techniques of global optimization. These authors further considered this problem by proposing parametric formulations and difference of two convex functions (DC) programming approaches. A similar approach can be found in [1]. However, finding a global optimum in optimization with BMI constraints is in general impractical and global optimization methods are usually recommended only for low dimensional problems. Our method developed in this paper is classified as a local optimization method which aims to find a local optimum based on solving a sequence of convex semidefinite programming problems. The approach in this paper is to generalize the idea of DC programming to optimization with convex-concave matrix inequality constraints. However, this is not only a technical extension since many characterizations of standard nonlinear programming are no longer preserved in nonlinear semidefinite programming, see, e.g. [25], [28]. Moreover, converting a nonlinear semidefinite programming problem into a standard nonlinear programming one usually requires some spectral functions which are related to the eigenvalues of matrix mappings. The resulting problem is in general nonconvex and nonsmooth, see, e.g. [7].

Sequential semidefinite programming method for nonlinear SDP and its application to robust control was considered by Fares *et al* in [9]. Thevenet *et al* [30] studied spectral SDP methods for solving problems involving BMI arising in controller design. Another approach is based on the fact that problems with BMI constraints can be reformulated as problems with LMI constraints and additional rank constraints. In [22] Orsi *et al* developed a Newton-like method for solving problems of this type.

In this paper, we are interested in optimization problems arising in static output feedback controller design for a linear, time-invariant system of the form:

$$\begin{cases} \dot{x} = Ax + B_1 w + Bu, \\ z = C_1 x + D_{11} w + D_{12} u, \\ y = Cx + D_{21} w, \end{cases} \quad (1)$$

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where $x \in \mathbb{R}^n$ is the state vector, $w \in \mathbb{R}^{n_w}$ is the performance input, $u \in \mathbb{R}^{n_u}$ is the input vector, $z \in \mathbb{R}^{n_z}$ is the performance output, $y \in \mathbb{R}^{n_y}$ is the physical output vector, $A \in \mathbb{R}^{n \times n}$ is state matrix, $B \in \mathbb{R}^{n \times n_u}$ is input matrix and $C \in \mathbb{R}^{n_y \times n}$ is the output matrix. Using a static feedback controller of the form $u = Fy$ with $F \in \mathbb{R}^{n_u \times n_y}$, we can write the closed-loop system as follows:

$$\begin{cases} \dot{x}_F = A_F x_F + B_F w, \\ z = C_F x_F + D_F w. \end{cases} \quad (2)$$

The stabilization, \mathcal{H}_2 , \mathcal{H}_∞ optimization and other control problems for this closed-loop system will be considered.

Contribution. Many control problems can be expressed as optimization problems with BMI constraints and these optimization problems can conveniently be reformulated as optimization problems of difference of two *positive semidefinite convex* (psd-convex) mappings (or convex-concave decomposition) constraints (see Definition 2.1 below). In this paper, we propose to use this reformulation leading to a new local optimization method for solving some classes of optimization problems involving BMI constraints. We provide a practical algorithm and prove the convergence of the algorithm under certain standard assumptions.

The algorithm proposed in this paper is very simple to implement by using available SDP software tools. Moreover, it does not require any globalization strategy such as line-search procedures to guarantee global convergence to a local minimum. The method still works in practice for nonsmooth optimization problems, where the objective function and the concave parts are only subdifferentiable, but not necessarily differentiable. Note that our method is different from the standard DCA approach in [24], [26] since we work directly with positive semidefinite matrix inequality constraints instead of transforming into DC representations as in [1], [2].

We show that our method is applicable to many control problems in static state/output feedback controller design. The numerical results are benchmarked using the data from the COMPl_eib library. Note, however, that this method is also applicable to other nonconvex optimization problems with matrix inequality constraints which can be written as a convex-concave decomposition.

Outline of the paper. The remainder of the paper is organized as follows. Section II provides some preliminary results which will be used in what follows. Section III presents the formulation of optimization problems involving convex-concave matrix inequality constraints and a fundamental assumption, Assumption A1. The algorithm and its convergence results are presented in Section IV. Applications to control problems on static feedback controller design and numerical benchmarking are given in Section V. The last section contains some concluding remarks.

II. PRELIMINARIES

Let \mathcal{S}^p be the set of symmetric matrices of size $p \times p$, \mathcal{S}_+^p , and resp., \mathcal{S}_{++}^p be the set of symmetric positive semidefinite, resp., positive definite matrices. For given matrices X and Y in \mathcal{S}^p , the relation $X \succeq Y$ (resp., $X \preceq Y$) means that $X - Y \in \mathcal{S}_+^p$ (resp., $Y - X \in \mathcal{S}_+^p$) and $X \succ Y$ (resp.,

$X \prec Y$) is $X - Y \in \mathcal{S}_{++}^p$ (resp., $Y - X \in \mathcal{S}_{++}^p$). The quantity $X \circ Y := \text{trace}(X^T Y)$ is an inner product of two matrices X and Y defined on \mathcal{S}^p , where $\text{trace}(Z)$ is the trace of matrix Z .

Definition 2.1: [25] A matrix-valued mapping $G : \mathbb{R}^n \rightarrow \mathcal{S}^p$ is said to be positive semidefinite convex (*psd-convex*) on a convex subset $C \subseteq \mathbb{R}^n$ if for all $t \in [0, 1]$ and $x, y \in C$, one has

$$G(tx + (1-t)y) \preceq tG(x) + (1-t)G(y). \quad (3)$$

If (3) holds true for \prec instead of \preceq for $t \in (0, 1)$ then G is said to be *strictly psd-convex* on C . Alternatively, if we replace \preceq in (3) by \succeq then G is said to be psd-concave on C . It is obvious that any convex function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is psd-convex with $p = 1$.

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be *strongly convex* with parameter $\rho > 0$ if $f(\cdot) - \frac{1}{2}\rho\|\cdot\|^2$ is convex.

The derivative of a matrix-valued mapping G at x is a linear mapping DG from \mathbb{R}^n to $\mathbb{R}^{p \times p}$ which is defined by

$$DG(x)h := \sum_{i=1}^n h_i \frac{\partial G}{\partial x_i}(x), \quad \forall h \in \mathbb{R}^n.$$

For a given convex set $X \in \mathbb{R}^n$, the matrix-valued mapping G is said to be differentiable on a subset X if its derivative $DG(x)$ exists at every $x \in X$. The definitions of the second order derivatives of matrix-valued mappings can be found, e.g., in [25]. Let $A : \mathbb{R}^n \rightarrow \mathcal{S}^p$ be a linear mapping defined as $Ax := \sum_{i=1}^n x_i A_i$, where $A_i \in \mathcal{S}^p$ for $i = 1, \dots, n$. The adjoint operator of A , A^* , is defined as $A^*Z := (A_1 \circ Z, A_2 \circ Z, \dots, A_n \circ Z)^T$ for any $Z \in \mathcal{S}^p$.

Lemma 2.2: a) A matrix-valued mapping G is psd-convex on X if and only if for any $v \in \mathbb{R}^p$ the function $\varphi(x) := v^T G(x)v$ is convex on X .

b) A mapping G is psd-convex on X if and only if for all x and y in X , one has

$$G(y) - G(x) \succeq DG(x)(y - x). \quad (4)$$

Proof: The proof of the statement a) can be found in [25]. We prove b). Let $\varphi(x) = v^T G(x)v$ for any $v \in \mathbb{R}^p$. If G is psd-convex then φ is convex. We have $\varphi(y) - \varphi(x) \geq \nabla \varphi(x)^T (y - x)$. Now, $\nabla \varphi(x)^T (y - x) = \sum_{i=1}^n (y_i - x_i) v^T \frac{\partial G}{\partial x_i}(x) v = v^T [DG(x)(y - x)]v$. Hence, $v^T [G(y) - G(x) - DG(x)(y - x)]v \geq 0$ for all v . We conclude that (4) holds. Conversely, if (4) holds then, for any v , we have $v^T [G(y) - G(x) - DG(x)(y - x)]v \geq 0$, which is equivalent to $\varphi(y) - \varphi(x) \geq \nabla \varphi(x)^T (y - x)$. Thus φ is convex. By virtue of a), the mapping G is psd-convex. ■

For simplicity of discussion, throughout this paper, we assume that all the functions and matrix-valued mappings are *twice differentiable* on their domain [25], [30]. However, this assumption can be reduced to the *subdifferentiability* of the objective function and the concave parts of the convex-concave decompositions of the matrix-valued mappings as in Definition 2.3 below.

Definition 2.3: A matrix-valued mapping $F : \mathbb{R}^n \rightarrow \mathcal{S}^p$ is said to be a *psd-convex-concave* mapping if F can be represented as a difference of two psd-convex mappings, i.e. $F(x) = G(x) - H(x)$, where G and H are psd-convex.

The pair (H, G) is called a psd-DC (or psd-convex-concave) decomposition of F .

Note that each given psd-convex-concave mapping possesses many psd-convex-concave decompositions.

III. OPTIMIZATION OF CONVEX-CONCAVE MATRIX INEQUALITY CONSTRAINTS

A. Psd-convex-concave decomposition of BMIs

Instead of using the vector x as a decision variable, we use from now on the matrix X as a matrix variable in $\mathbb{R}^{m \times n}$. Note that any matrix X can be considered as an $m \times n$ -column vector by vectorizing with respect to its columns, i.e. $x = \text{vec}(X) := (X_{11}, X_{21}, \dots, X_{mn})^T$. The inverse mapping of vec is called mat . Since vec and mat are linear operators, the psd-convexity is still preserved under these operators.

A mapping $F : \mathbb{R}^{p \times q} \times \mathcal{S}^p \rightarrow \mathcal{S}^p$ given by $F(X, Y) := XQ^{-1}X^T - Y$, where $Q \in \mathcal{S}_{++}^q$, is called a *Schur psd-convex*¹ mapping.

Let us consider a bilinear matrix form

$$F(X, Y) := X^T Y + Y^T X. \quad (5)$$

By using the Kronecker product, we can write F as $\text{vec}(F(X, Y)) = (I_n \otimes X^T)\text{vec}(Y) + (I_y \otimes Y^T)\text{vec}(X) = (\sum_{i,j} x_i y_j)$, where I_n, I_y are appropriate identity matrices, \otimes denotes the Kronecker product. Hence, the vectorization of $F(X, Y)$ is indeed a bilinear form of two vectors $x := \text{vec}(X)$ and $y := \text{vec}(Y)$.

The following lemma shows that the bilinear matrix form (5) can be decomposed as a difference of two psd-convex mappings.

Lemma 3.1: a) The mappings $f(X) := X^T X$ and $g(X) := X X^T$ are psd-convex on $\mathbb{R}^{m \times n}$. The mapping $f(X) := X^{-1}$ is psd-convex on \mathcal{S}_{++}^p .

b) The bilinear matrix form $b(X, Y) := X^T Y + Y^T X$ can be represented as a psd-convex-concave mapping of at least three forms:

$$\begin{aligned} b(X, Y) &= (X + Y)^T (X + Y) - (X^T X + Y^T Y) \\ &= X^T X + Y^T Y - (X - Y)^T (X - Y) \quad (6) \\ &= \frac{1}{2}[(X + Y)^T (X + Y) - (X - Y)^T (X - Y)]. \end{aligned}$$

The statement b) provides at least three different explicit psd-convex-concave decompositions of the bilinear form $b(X, Y)$. Intuitively, we can see that the first decomposition has a ‘‘strong curvature’’ on the second term, while the second and the third decompositions have ‘‘less curvature’’ on the second term due to the compensation between X and Y .

The following result will be used to transform Schur psd-convex constraints to LMI constraints.

Lemma 3.2: a) Suppose that $A \in \mathcal{S}^n$. Then the matrix inequality $BB^T - A \prec (\preceq) 0$ is equivalent to

$$\begin{bmatrix} A & B \\ B^T & I \end{bmatrix} \succ (\succeq) 0. \quad (7)$$

b) Suppose that $A \in \mathcal{S}^n, D \succ 0$, then we have:

$$\begin{bmatrix} A - BB^T & C \\ C^T & D \end{bmatrix} \succ (\succeq) 0 \iff \begin{bmatrix} A & B & C \\ B^T & I & O \\ C^T & O & D \end{bmatrix} \succ (\succeq) 0. \quad (8)$$

The proof of this lemma immediately follows by applying Schur’s complement and Lemma 2.2 [6]. We omit the proof here.

B. Optimization involving convex-concave matrix inequality constraints

Let us consider the following optimization problem:

$$\begin{cases} \min_x & f(x) \\ \text{s.t.} & G_i(x) - H_i(x) \preceq 0, \quad i = 1, \dots, l, \\ & x \in \Omega, \end{cases} \quad (9)$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex, $\Omega \subseteq \mathbb{R}^n$ is a nonempty, closed convex set, and G_i and H_i ($i = 1, \dots, l$) are psd-convex. Problem (9) is referred to as a convex optimization problem with psd-convex-concave matrix inequality constraints.

Let Ω be a polyhedral in \mathbb{R}^n . Then, if f is nonlinear or one of the mappings G_i or H_i ($i = 1, \dots, l$) is nonlinear then (9) is a nonlinear semidefinite program. If H_i ($i = 1, \dots, l$) are linear then (9) is a convex nonlinear SDP problem. Otherwise, it is a nonconvex nonlinear SDP problem.

Let us define $\mathcal{L}(x, \Lambda) := f(x) + \sum_{i=1}^l \Lambda_i \circ [G_i(x) - H_i(x)]$ as the Lagrange function of (9), where $\Lambda_i \in \mathcal{S}^p$ ($i = 1, \dots, l$) considered as Lagrange multipliers. The generalized KKT condition of (9) is presented as:

$$\begin{cases} 0 \in \nabla f(x) + \sum_{i=1}^l D[G_i(x) - H_i(x)]^* \Lambda_i + \mathcal{N}_\Omega(x), \\ G_i(x) - H_i(x) \preceq 0, \quad \Lambda_i \succeq 0, \\ [G_i(x) - H_i(x)] \circ \Lambda_i = 0, \quad i = 1, \dots, l. \end{cases} \quad (10)$$

Here, $\mathcal{N}_\Omega(x)$ is the normal cone of Ω at x defined as

$$\mathcal{N}_\Omega(x) := \begin{cases} \{w \in \mathbb{R}^n \mid w^T(x - y) \geq 0, \forall y \in \Omega\}, & \text{if } x \in \Omega, \\ \emptyset, & \text{otherwise.} \end{cases}$$

A pair (x^*, Λ^*) satisfying (10) is called a KKT point, x^* is called a stationary point and Λ^* is the corresponding multiplier of (9). The generalized optimality condition for nonlinear semidefinite programming can be found in literature, e.g., [25], [28].

Let us denote by

$$\mathcal{D} := \{x \in \Omega \mid G_i(x) - H_i(x) \preceq 0, \quad i = 1, \dots, l\}, \quad (11)$$

the feasible set of (9) and by $\text{ri}(\mathcal{D})$ the relative interior of \mathcal{D} which is defined by

$$\text{ri}(\mathcal{D}) := \{x \in \text{ri}(\Omega) \mid G_i(x) - H_i(x) \prec 0, \quad i = 1, \dots, m\},$$

where $\text{ri}(\Omega)$ is the set of classical relative interiors of Ω [6]. The following condition is a fundamental assumption in this paper.

Assumption A1: $\text{ri}(\mathcal{D})$ is nonempty.

Note that this assumption is crucial for our method, because, as we shall see, it requires a strictly feasible starting point $x^0 \in \text{ri}(\mathcal{D})$. Finding such a point is in principle not an easy task. However, in many problems, this assumption is always satisfied. In Section V we will propose techniques to determine a starting point for the control problems under consideration.

¹Due to Schur’s complement form

IV. THE ALGORITHM AND ITS CONVERGENCE

In this section, a local optimization method for finding a stationary point of problem (9) is proposed. Motivated from the DC programming algorithm developed in [24] and the convex-concave procedure in [26] for scalar functions, we develop an iterative procedure for finding a stationary point of (9). The main idea is to linearize the nonconvex part of the psd-convex-concave matrix inequality constraints and then transform the linearized subproblem into a quadratic semidefinite programming problem. The subproblem can be either directly solved by means of interior point methods or transformed into a quadratic problem with LMI constraints. In the latter case, the resulting problem can be solved by available software tools such as Sedumi [27] and SDPT3 [29].

A. The algorithm

Suppose that $x^k \in \Omega$ is a given point, the linearized problem of (9) around x^k is written as

$$\begin{cases} \min_x & \{f_k(x) := f(x) + \frac{\rho_k}{2} \|Q_k(x - x_k)\|_2^2\} \\ \text{s.t.} & G_i(x) - H_i(x^k) - DH_i(x^k)(x - x^k) \preceq 0, \quad i = 1, \dots, l, \\ & x \in \Omega. \end{cases} \quad (12)$$

Here, we add a regularization term into the objective function of the original problem, where Q_k is a given matrix that projects $x - x_k$ in a certain subspace of \mathbb{R}^n and $\rho_k \geq 0$ is a regularization parameter. Since G_i ($i = 1, \dots, l$) are psd-convex and the objective function is convex, problem (12) is convex. The *linearized convex-concave SDP algorithm* for solving (9) is described as follows.

ALGORITHM 1:

Initialization: Choose a positive number ρ_0 and a matrix $Q_0 \in S_+^n$. Find an initial point $x^0 \in \text{ri}(\mathcal{D})$. Set $k := 0$.

Iteration k : For $k = 0, 1, \dots$. Perform the following steps:

Step 1: Solve the convex semidefinite program (12) to obtain a solution x^{k+1} .

Step 2: If $\|x^{k+1} - x^k\| \leq \varepsilon$ for a given tolerance $\varepsilon > 0$ then terminate. Otherwise, update ρ_k and Q_k (if necessary), set $k := k + 1$ and go back to *Step 1*.

The following main property of the method makes an implementation very easy. If the initial point x^0 belongs to the relative interiors of the feasible set \mathcal{D} , i.e. $x^0 \in \text{ri}(\mathcal{D})$, then Algorithm 1 generates a sequence x^k which still belongs to \mathcal{D} . In particular, no line-search procedure is needed to ensure global convergence.

This property follows from the fact that the linearization of the concave part $-H_i$ is its an overestimate of this mapping (in the sense of the positive semidefinite cone), i.e.

$$-H_i(x) \preceq -H_i(x^k) - DH_i(x^k)(x - x^k), \quad \forall x \in \Omega,$$

which is equivalent to

$$G_i(x) - H_i(x) \preceq G_i(x) - H_i(x^k) - DH_i(x^k)(x - x^k), \quad \forall x \in \Omega.$$

Hence, if the subproblem (12) has a solution x^{k+1} then it is feasible to (9). Geometrically, Algorithm 1 can be seen as an inner approximation method.

The main tasks of an implementation of Algorithm 1 consist of:

- determining an initial point $x^0 \in \text{ri}(\mathcal{D})$, and
- solving the convex semidefinite program (12) repeatedly.

As mentioned before, since \mathcal{D} is nonconvex, finding an initial point x^0 in $\text{ri}(\mathcal{D})$ is, in principle, not an easy task. However, in some practical problems, this can be done by exploiting the special structure of the problem (see the examples in Section V).

To solve the convex subproblem (12), we can either implement an interior point method and exploit the structure of the problem or transform it into a standard SDP problem and then make use of available software tools for SDP. The regularization parameter ρ_k and the projection matrix Q_k can be fixed at appropriate choices for all iterations, or adaptively updated.

Lemma 4.1: If x^k is a solution of (12) linearized at x^k then it is a stationary point of (9).

Proof: Suppose that Λ^{k+1} is a multiplier associated with x^k , substituting x^k into the generalized KKT condition (39) of (12) we obtain (10). Thus x^k is a stationary point of (9). ■

B. Convergence analysis

In this subsection, we restrict our discussion to the following special case.

Assumption A2: The mappings G_i ($i = 1, \dots, l$) are Schur psd-convex and Ω is formed by a finite number of LMIs. In addition, f is convex quadratic on \mathbb{R}^n with a convexity parameter $\rho_f \geq 0$.

This assumption is only technical for our implementation. If the mapping G_i is Schur psd-convex then the linearized constraints of problem (12) can directly be transformed into LMI constraints (see Lemma 3.2). In practice, G_i ($i = 1, \dots, l$) can be general psd-convex mappings and f can be a general convex function.

Under Assumption A2, the convex subproblem (12) can be transformed equivalently into a quadratic semidefinite program of the form:

$$\begin{cases} \min_{z \in \mathbb{R}^{n_z}} & \frac{1}{2} z^T B z + h^T z \\ \text{s.t.} & A(z) + C \preceq 0, \end{cases} \quad (13)$$

where A is a linear mapping from \mathbb{R}^{n_z} to S^{p_z} , $C \in S^{p_z}$ and B is a symmetric matrix, by means of Lemma 3.2.

A vector \hat{z} is said to satisfy the *Slater condition* of (13) if $A(\hat{z}) + C \prec 0$. Suppose that the triple $(\bar{z}, \bar{V}, \bar{S})$ satisfies the KKT condition of (13) (see [10]), where \bar{z} is a primal stationary point, \bar{V} is a Lagrange multiplier and \bar{S} is a slack variable associated with \bar{z} and \bar{V} . Then, problem (13) is said to satisfy the *strict complementarity* condition at $(\bar{z}, \bar{V}, \bar{S})$ if $\bar{V} + \bar{S} \succ 0$.

Let \bar{z} be a stationary point of (13). We say that $0 \neq p \in \mathbb{R}^{n_z}$ is a *feasible direction* to (13) if $\bar{z} + \varepsilon p$ is a feasible point of (13) for all $\varepsilon > 0$ sufficiently small. As in [10], we assume that the *second order sufficient condition* holds for (13) at \bar{z} with modulus $\mu > 0$ if for all feasible directions p at \bar{z} with $p^T(h + B\bar{z}) = 0$, one has $p^T B p \geq \mu \|p\|^2$. We say that the convex

problem (13) is solvable and satisfies the *strong second order sufficient condition* if there exists a KKT point $(\bar{z}, \bar{V}, \bar{S})$ of the KKT system of (13) that satisfies the second order sufficient condition and the strict complementary condition.

Assumption A3: The convex subproblem (12) is *solvable* and satisfies the *strong second order sufficient condition*.

Assumption A3 is standard in optimization and is usually used to investigate the convergence of the algorithms [9], [10], [25].

The following lemma shows that $\Delta x^k := x^{k+1} - x^k$ is a descent direction of problem (9) whose proof can be found in the appendix.

Lemma 4.2: Suppose that $\{(x^k, \Lambda^k)\}_{k \geq 0}$ is a sequence generated by Algorithm 1. Then:

a) The following inequality holds for $k \geq 0$:

$$f(x^{k+1}) - f(x^k) \leq -\frac{\rho_f}{2} \|x^{k+1} - x^k\|_2^2 - \rho_k \|Q_k(x^{k+1} - x^k)\|_2^2, \quad (14)$$

where ρ_f is the convexity parameter of f .

b) If there exists at least one constraint $i_0, i_0 \in \{1, 2, \dots, l\}$, to be strictly feasible at x^k , i.e. $G_{i_0}(x^k) - H_{i_0}(x^k) < 0$, then $f(x^{k+1}) < f(x^k)$ provided that $\Lambda_{i_0}^{k+1} > 0$.

c) If $\rho_k > 0$ and Q_k is full-row-rank then Δx^k is a sufficient descent direction of (9), i.e. $f(x^{k+1}) - f(x^k) \leq -\rho_k \|Q_k \Delta x^k\|_2^2 < 0$ for all $k \geq 0$.

The following theorem shows the convergence of Algorithm 1 in a particular case.

Theorem 4.3: Under Assumptions A1, A2 and A3, suppose that f is bounded from below on \mathcal{D} , where \mathcal{D} is assumed to be bounded in \mathbb{R}^n . Let $\{(x^k, \Lambda^k)\}$ be a sequence generated by Algorithm 1 starting from $x^0 \in \text{ri}(\mathcal{D})$. Then if either f is strongly convex or $\rho_k \equiv \rho > 0$ and $Q_k \equiv Q$ is full-row-rank for all $k \geq 0$ then every accumulation point (x^*, Λ^*) of $\{(x^k, \Lambda^k)\}$ is a KKT point of (9). Moreover, if the set of the KKT points of (9) is finite then the whole sequence $\{(x^k, \Lambda^k)\}$ converges to a KKT point of (9).

Proof: Let $\mathcal{M}(x^0) := \{x^k \mid k \geq 0\}$ be the sequence of sample points generated by Algorithm 1 starting from x^0 . For a given $x \in \Omega$, let us define the following mapping:

$$\mathcal{A}_{\text{sol}}(x) := \underset{y \in \Omega}{\text{argmin}} \left\{ f(y) + \frac{\rho}{2} \|Q(y-x)\|_2^2 \mid y \in \Omega, \right. \\ \left. G_i(y) - H_i(x) - DH_i(x)(y-x) \leq 0, \quad i=1, \dots, m \right\}. \quad (15)$$

Then, \mathcal{A}_{sol} is a multivalued mapping and it can be considered as the solution mapping of the convex subproblem (12). Note that the sequence $\{x^k\}_{k \geq 0}$ generated by Algorithm 1 satisfies $x^{k+1} \in \mathcal{A}_{\text{sol}}(x^k)$ for all $k \geq 0$. We first prove that \mathcal{A}_{sol} is a closed mapping. Indeed, since the convex subproblem (12) satisfies Slater's condition and has a solution that satisfies the strict complementarity and the second order sufficient condition, by applying Theorem 1 in [10] we conclude that the mapping \mathcal{A}_{sol} is differentiable in a neighborhood of the solution. In particular, it is closed due to the compactness of \mathcal{D} .

On the other hand, since f is either strongly convex or $\rho_k \equiv \rho > 0$ for all $k \geq 0$ and $Q_k \equiv Q$ is full-row-rank, it follows from Lemma 4.2 that the objective function f is

strictly monotone on $\mathcal{M}(x^0)$, i.e. $f(x^{k+1}) < f(x^k)$ for all $x^k, x^{k+1} \in \mathcal{M}(x^0)$. Since $\mathcal{M}(x^0) \subseteq \mathcal{D}$ and \mathcal{D} is compact, $\mathcal{M}(x^0)$ is also compact. Applying Theorem 2 in [20] we conclude that every limit points of the sequence $\{x^k\}_{k \geq 0}$ belongs to the set of stationary points S^* . Moreover, since f is bounded from below and either f is strongly convex or $\rho > 0$ and Q is full-row rank, it follows from (14) that $\lim_{k \rightarrow \infty} \|x^{k+1} - x^k\| = 0$. Therefore, S^* is connected and if S^* is finite then the whole sequence $\{x^k\}_{k \geq 0}$ converges to x^* in S^* . ■

Remark 4.4: The condition that f is quadratic in Assumption 2 can be relaxed to f being twice continuously differentiable. However, in this case, we need a direct proof for Theorem 4.3 instead of applying Theorem 1 in [10].

V. APPLICATIONS TO ROBUST CONTROLLER DESIGN

In this section, we apply the method developed in the previous sections to the following static state/output feedback controller design problems:

- 1) Sparse linear static output feedback controller design;
- 2) Spectral abscissa and pseudospectral abscissa optimization;
- 3) \mathcal{H}_2 optimization;
- 4) \mathcal{H}_∞ optimization;
- 5) and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis.

We used the system data from [13], [23] and the COMPlib library [17]. All the implementations are done in Matlab 7.11.0 (R2010b) running on a PC Desktop Intel(R) Core(TM)2 Quad CPU Q6600 with 2.4GHz and 3Gb RAM. We use the YALMIP package [19] as a modeling language and SeDuMi 1.1 as a SDP solver [27] to solve the LMI optimization problems arising in Algorithm 1 at the initial phase (Phase 1) and subproblem (12). We also benchmarked our method with various examples and compared our results with HIFOO [12] and PENBMI [14] for all control problems. HIFOO is an open-source Matlab package for fixed-order controller design. It computes a fixed-order controller using a hybrid algorithm for nonsmooth, nonconvex optimization based on quasi-Newton updating and gradient sampling. PENBMI [14] is a commercial software for solving optimization problems with quadratic objective and BMI constraints, which is freely licensed for academic purposes. We initialized the initial controller for HIFOO and the BMI parameters for PENBMI to the initial values of our method. As shown in [22], we can reformulate the spectral abscissa feasibility problem as a rank constrained LMI feasibility problem. Therefore, we also compared our results with LMIRank [22] (a MATLAB toolbox for solving rank constrained LMI feasibility problems) by implementing a simple procedure for solving the spectral abscissa optimization.

Note that all problems addressed here lead to at least one BMI constraint. To apply the method developed in the previous sections, we propose a unified scheme to treat these problems.

SCHEME A.1:

- Step 1.* Find a convex-concave decomposition of the BMI constraints as $G(x) - H(x) \leq 0$.
- Step 2.* Find a starting point $x^0 \in \text{ri}(\mathcal{D})$.

Step 3. For a given x^k , linearize the concave part to obtain the convex constraint $G(x) - H_k(x) \preceq 0$, where H_k is the linearization of H at x^k .

Step 4. Reformulate the convex constraint as an LMI constraint by means of Lemma 3.2.

Step 5. Apply Algorithm 1 with an SDP solver to solve the given problem.

A. Sparse linear constant output-feedback design

Let us consider a BMI optimization problem of sparse linear constant output-feedback design given as:

$$\begin{aligned} \min_{\alpha, P, F} \quad & \{f(\alpha, P, F) := -\sigma\alpha + \sum_{i=1}^{n_u} \sum_{j=1}^{n_y} |F_{ij}|\} \\ \text{s.t.} \quad & (A+BFC)^T P + P(A+BFC) + 2\alpha P \prec 0, \\ & P = P^T, P \succ 0. \end{aligned} \quad (16)$$

Here, matrices A, B, C are given with appropriate dimensions, P and F are referred to as variables and $\sigma > 0$ is a weighting parameter. The objective function consists of two terms: the first term $\sigma\alpha$ is to stabilize the system (or to maximize the decay rate) and the second one is to ensure the sparsity of the gain matrix F . This problem is a modification of the first example in [13]. Let us illustrate SCHEME A.1 for solving this problem.

Step 1: Let $B_F := A + BFC + \alpha I$, where I is the identity matrix. Then, applying Lemma 3.1 we can write

$$(A+BFC)^T P + P(A+BFC) + 2\alpha P = B_F^T P + P B_F \\ = B_F^T B_F + P^T P - (B_F - P)^T (B_F - P), \quad (17)$$

$$= \frac{1}{2} [(B_F + P)^T (B_F + P) - (B_F - P)^T (B_F - P)]. \quad (18)$$

In our implementation, we use the decomposition (18).

If we denote by

$$\text{and} \quad \begin{aligned} G(\alpha, P, F) &:= \frac{1}{2} (B_F + P)^T (B_F + P), \\ H(\alpha, P, F) &:= \frac{1}{2} (B_F - P)^T (B_F - P), \end{aligned} \quad (19)$$

then the BMI constraint in (16) can be written equivalently as a psd-concave matrix inequality constraint (of a variable x formed from (α, P, F) as $x := (\alpha, \text{vec}(P)^T, \text{vec}(F)^T)^T$) as follows:

$$G(\alpha, P, F) - H(\alpha, P, F) \prec 0. \quad (20)$$

Note that the objective function of (16) is convex but nonsmooth which is not directly suitable for the sequential SDP approach in [8], but, the nonconvex problem (16) can be reformulated in the form of (9) by using slack variables.

Steps 2-5: The implementation is carried out as follows:

Phase 1. (Determine a starting point $x^0 \in \text{ri}(\mathcal{D})$). Set $F^0 := 0$, $\alpha^0 := -\alpha_0(A^T + A)/2$ where $\alpha_0(\cdot)$ is the maximum real part of the eigenvalues of the matrix, and compute $P = P^0$ as the solution of the LMI feasibility problem

$$(A + BF^0 C)^T P + P(A + BF^0 C) + 2\alpha^0 P \prec 0. \quad (21)$$

The above choice for (α^0, F^0) originates from the property that $P^0 = I$ renders the left hand side of (21) negative semidefinite (but not negative definite).

Phase 2. Perform Algorithm 1 with a starting point x^0 found at Phase 1.

Let us now illustrate *Step 4* of SCHEME A.1. After linearizing the concave part of the convex-concave reformulation of the last BMI constraint in (16) at (F^k, P^k, α^k) we obtain the linearization:

$$(A+BFC+\alpha I+P)^T (A+BFC+\alpha I+P) - H_k(F, P, \alpha) \prec 0, \quad (22)$$

where $H_k(F, P, \alpha)$ is a linear mapping of F, P and α . Now, by applying Lemma 3.2, (22) can be transformed into an LMI constraint:

$$\begin{bmatrix} H_k(F, P, \alpha) & (A+BFC+\alpha I+P)^T \\ (A+BFC+\alpha I+P) & I \end{bmatrix} \succ 0.$$

With the above approach we solved problem (16) for the same system data as in [13]. Here, matrices A, B and C are given, respectively as:

$$A = \begin{bmatrix} 2.45 & -0.90 & 1.53 & -1.26 & 1.76 \\ -0.12 & -0.44 & -0.01 & 0.69 & 0.90 \\ 2.07 & -1.20 & -1.14 & 2.04 & -0.76 \\ -0.59 & 0.07 & 2.91 & -4.63 & -1.15 \\ -0.74 & -0.23 & -1.19 & -0.06 & -2.52 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.81 & -0.79 & 0.00 & 0.00 & -0.95 \\ -0.34 & -0.50 & 0.06 & 0.22 & 0.92 \\ -1.32 & 1.55 & -1.22 & -0.77 & -1.14 \\ -2.11 & 0.32 & 0.00 & -0.83 & 0.59 \\ 0.31 & -0.19 & -1.09 & 0.00 & 0.00 \end{bmatrix},$$

$$\text{and } C = \begin{bmatrix} 0.00 & 0.00 & 0.16 & 0.00 & -1.78 \\ 1.23 & -0.38 & 0.75 & -0.38 & -0.00 \\ 0.46 & 0.00 & -0.05 & 0.00 & 0.00 \\ 0.00 & -0.12 & 0.23 & -0.12 & 1.14 \end{bmatrix}.$$

The weighting parameter σ is chosen by $\sigma = 3$. Algorithm 1 is terminated if one of the following conditions is satisfied:

- subproblem (12) encounters a numerical problem;
- $\|\Delta x^k\|_\infty / (\|x^k\|_\infty + 1) \leq 10^{-3}$;
- the maximum number of iterations, K_{\max} , reaches;
- or the objective function is not significantly improved after two successive iterations, i.e. $|f^{k+1} - f^k| \leq 10^{-4}(1 + |f^k|)$ for some $k = \bar{k}$ and $k = \bar{k} + 1$, where $f^k := f(x^k)$.

In this example, Algorithm 1 is terminated after 15 iterations, whereas the objective function is not significantly improved. However, after the 2nd iteration, matrix F only has 3 nonzero elements, while the decay rate α is 1.17316. This value is much higher than the one reported in [13], $\alpha = 0.3543$ after 6 iterations. We obtain the gain matrix F as

$$F = \begin{bmatrix} 0.6540 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & -0.4872 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 1.1280 & 0.0000 & 0.0000 \end{bmatrix}.$$

With this matrix, the maximum real part of the eigenvalues of the closed-loop matrix in (2), $A_F := A + BFC$, is $\alpha_0(A_F) := -1.40706$. Simultaneously, $\alpha_0(A_F^T P + P A_F + 2\alpha P) = -0.327258 < 0$ and $\alpha_0(P) = 0.587574 > 0$. Note that $\alpha_0(A_F) \neq -\alpha$ due to the in-activeness of the BMI constraint in (16) at the 2nd iteration.

B. Spectral abscissa and pseudo-spectral abscissa optimization

One popular problem in control theory is to optimize the spectral abscissa of the closed-loop system $\dot{x} = (A + BFC)x$. Briefly, this problem is presented as an unconstrained optimization problem of the form:

$$\min_{F \in \mathbb{R}^{n_u \times n_y}} \alpha_0(A + BFC), \quad (23)$$

where $\alpha_0(A + BFC) := \sup \{\operatorname{Re}(\lambda) \mid \lambda \in \Lambda(A + BFC)\}$ is the spectral abscissa of $A + BFC$, $\operatorname{Re}(\lambda)$ denotes the real part of $\lambda \in \mathbb{C}$ and $\Lambda(A + BFC)$ is the spectrum of $A + BFC$. Problem (23) has many drawbacks in terms of numerical solution due to the nonsmoothness and non-Lipschitz continuity of the objective function α_0 [7].

In order to apply the method developed in this paper, problem (23) is reformulated as an optimization problem with BMI constraints of the form, see, e.g. [7], [18]:

$$\begin{cases} \max_{P, F, \beta} \beta \\ \text{s.t.} \quad (A + BFC)^T P + P(A + BFC) + 2\beta P \prec 0, \\ \quad \quad P = P^T, P \succ 0. \end{cases} \quad (24)$$

Here, matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_u}$ and $C \in \mathbb{R}^{n_y \times n}$ are given. Matrices $P \in \mathbb{R}^{n \times n}$ and $F \in \mathbb{R}^{n_u \times n_y}$ and the scalar β are considered as variables. If the optimal value of (24) is strictly positive then the closed-loop feedback controller $u = Fy$ stabilizes the linear system $\dot{x} = (A + BFC)x$.

Problem (24) is very similar to (16). Therefore, using the same trick as in (16), we can reformulate (24) in the form of (9). More precisely, if we define $B_F := A + BFC + \beta I$ then the bilinear matrix mapping $A_F^T P + P A_F$ can be represented as a psd-convex-concave decomposition of the form (18) and problem (24) can be rewritten in the form of (9). We implement Algorithm 1 for solving this resulting problem using the same parameters and the stopping criterions as in Subsection V-A. In addition, we regularize the objective function by adding the term $\frac{\rho_F}{2} \|F - F^k\|_F^2 + \frac{\rho_P}{2} \|P - P^k\|_F^2$, with $\rho_F = \rho_P = 10^{-2}$. The maximum number of iterations K_{\max} is set to 150.

We test for several problems in COMPl_{ib} and compare our results with the ones reported by HIFOO, PENBMI and LMIRank. For LMIRank, we implement the algorithm proposed in [22]. We initialize the value of the decay rate α at 10^{-4} and perform an iterative loop to increase α as $\alpha^{k+1} := \alpha^k + 0.1$. The algorithm is terminated if either the problems (12) or (21) in [22] with a correspondence α can not be solved or the maximum number of iterations $K_{\max} := 100$ is reached. The numerical results of four algorithms are reported in Table I. Here, we initialize the algorithm in HIFOO with the same initial guess $F^0 = 0$. Since PENBMI and our methods solve the same BMI problems, they are initialized by the same initial values for P , F and β .

The notation in Table I consists of: Name is the name of problems, $\alpha_0(A)$, $\alpha_0(A_F)$ are the maximum real part of the eigenvalues of the open-loop and closed-loop matrices A , A_F , respectively; iter is the number of iterations, time[s] is the CPU time in seconds. The columns titled HIFOO, LMIRank and PENBMI give the maximum real part of the

TABLE I
COMPUTATIONAL RESULTS FOR (24) IN COMPl_{ib}

Problem		Other Results, $\alpha_0(A_F)$			Results and Performances		
Name	$\alpha_0(A)$	HIFOO	LMIRANK	PENBMI	$\alpha_0(A_F)$	Iter	time[s]
AC1	0.000	-0.2061	-8.4766	-7.0758	-0.8535	41	12.44
AC4	2.579	-0.0500	-0.0500	-0.0500	-0.0500	14	4.60
AC5 ^a	0.999	-0.7746	-1.8001	-2.0438	-0.7389	28	63.33
AC7	0.172	-0.0322	-0.0204	0.0896	-0.0673	150	111.46
AC8	0.012	-0.1968	-0.4447	0.4447	-0.0755	24	21.95
AC9	0.012	-0.3389	-0.5230	-0.4450	-0.3256	78	74.57
AC11	5.451	-0.0003	-5.0577	-	-3.0244	61	38.44
AC12	0.580	-10.8645	-9.9658	-1.8757	-0.3414	150	86.72
HE1	0.276	-0.2457	-0.2071	-0.2468	-0.2202	150	87.64
HE3	0.087	-0.4621	-2.3009	-0.4063	-0.8702	47	48.80
HE4	0.234	-0.7446	-1.9221	-0.0909	-0.8647	63	71.66
HE5	0.234	-0.1823	-	-0.2932	-0.0587	150	178.71
HE6	0.234	-0.0050	-0.0050	-0.0050	-0.0050	12	41.00
REA1	1.991	-16.3918	-5.9736	-1.7984	-3.8599	77	79.23
REA2	2.011	-7.0152	-10.0292	-3.5928	-2.1778	40	39.15
REA3	0.000	-0.0207	-0.0207	-0.0207	-0.0207	150	362.21
DIS2	1.675	-6.8510	-10.1207	-8.3289	-8.4540	28	37.28
DIS4	1.442	-36.7203	-0.5420	-92.2842	-8.0989	72	124.23
WEC1	0.008	-8.9927	-8.7350	-0.9657	-0.8779	150	305.36
IH	0.000	-0.5000	-0.5000	-0.5000	-0.5000	7	39.41
CSE1	0.000	-0.4509	-0.4844	-0.4490	-0.2360	38	158.67
TF1	0.000	-	-	-0.0618	-0.1544	56	137.98
TF2	0.000	-	-	-1.0e-5	-1.0e-5	8	20.41
TF3	0.000	-	-	-0.0032	-0.0031	93	237.93
NN1	3.606	-3.0458	-4.4021	-4.3358	-0.8746	12	37.53
NN5 ^a	0.420	-0.0942	-0.0057	-0.0942	-0.0913	11	42.62
NN9	3.281	-2.0789	-0.7048	-	-0.0279	33	111.41
NN13	1.945	-3.2513	-4.5310	-9.0741	-3.4318	150	572.50
NN15	0.000	-6.9983	-11.0743	-0.0278	-0.8353	150	524.80
NN17	1.170	-0.6110	-0.5130	-	-0.6008	99	342.67

eigenvalues of the closed-loop system for a static output feedback controller computed by available software HIFOO [12], LMIRank [22] and PENBMI [14], respectively. Our results can be found in the sixth column. The entries with a dash sign indicate that there is no feasible solution found. Algorithm 1 fails or makes only slow progress towards a local solution with 6 problems: AC18, DIS5, PAS, NN6, NN7, NN12 in COMPl_{ib}. Problems AC5 and NN5 are initialized with a different matrix F^0 to avoid numerical problems.

Note that Algorithm 1 as well as the algorithms implemented in HIFOO, LMIRank and PENBMI are local optimization methods, which only report a local minimizer and these solutions may not be the same. Because the LMIRank package can only handle feasibility problems, it cannot directly be used to solve problem (24). Therefore, we have used a direct search procedure for finding α . The computational time of the overall procedure is much higher than the other methods for the majority of the test problems.

To conclude this subsection, we show that our method can also be applied to solve the problem of optimizing the pseudo-spectral abscissa in static feedback controller designs. This problem is described as follows (see [7], [18]):

$$\begin{cases} \max_{\beta, \mu, \omega, F, P} \beta \\ \text{s.t.} \quad \begin{bmatrix} 2\beta P + A_F^T P + P A_F + \mu I - \omega I_z & \varepsilon P \\ \varepsilon P & \omega I \end{bmatrix} \preceq 0, \\ \quad \quad P \succ 0, P = P^T, \mu > 0, \end{cases} \quad (25)$$

where $A_F = A + BFC$ as before and $\omega \leq 0$.

Using the same notation $B_F = A + BFC + \beta I$ as in (24) and applying the statement b) of Lemma 3.2, the BMI constraint in this problem can be transformed into a psd-convex-concave one:

$$\begin{bmatrix} B_F^T B_F + P^T P + (\mu - \omega) I & \varepsilon P \\ \varepsilon P & \omega I \end{bmatrix} - \begin{bmatrix} (B_F - P)^T (B_F - P) & 0 \\ 0 & 0 \end{bmatrix} \preceq 0.$$

If we denote the linearization of $(B_F - P)^T (B_F - P)$ at the iteration k by H_k , i.e. $H_k = (B_F - P)^T (B_F^k - P^k) + (B_F^k -$

$P^k)^T(B_F - P) - (B_F^k - P^k)^T(B_F^k - P^k)$, then the linearized constraint in the subproblem (12) can be represented as an LMI thanks to Lemma 3.2:

$$\begin{bmatrix} H_k + (\omega - \mu)I & B_F^T & P & -\varepsilon P \\ B_F & I & 0 & 0 \\ P & 0 & I & 0 \\ -\varepsilon P & 0 & 0 & -\omega I \end{bmatrix} \succeq 0.$$

Hence, Algorithm 1 can be applied to solve problem (25).

Remark 5.1: If we define $\bar{F} := BFC$ then the bilinear matrix mapping $A_F^T P + PA_F$ can be rewritten as

$$A_F^T P + PA_F = \frac{1}{2} [(P + \bar{F})^T (P + \bar{F}) - (P - \bar{F})^T (P - \bar{F})] - A^T P - PA.$$

Using this decomposition, one can avoid the contribution of matrix A on the bilinear term. Consequently, Algorithm 1 may work better in some specific problems.

C. \mathcal{H}_2 optimization: BMI formulation

In this subsection, we consider an optimization problem arising in \mathcal{H}_2 synthesis of the linear system (1). Let us assume that $D_{12} = 0$ and $D_{21} = 0$, then this problem is formulated as the following optimization problem with BMI constraints [17].

$$\begin{aligned} \min_{F, Q, X} \quad & \text{trace}(X) \\ \text{s.t.} \quad & (A + BFC)Q + Q(A + BFC)^T + B_1 B_1^T \preceq 0, \\ & \begin{bmatrix} X & C_1 Q \\ QC_1^T & Q \end{bmatrix} \succeq 0, \quad Q \succ 0. \end{aligned} \quad (26)$$

Here, we also assume that $B_1 B_1^T$ is positive definite. Otherwise, we use $B_1 B_1^T + \varepsilon I$ instead of $B_1 B_1^T$ with $\varepsilon = 10^{-5}$ in (26).

In order to apply Algorithm 1 for solving problem (26), a starting point $x^0 \in \text{ri}(\mathcal{D})$ is required. This task can be done by performing some extra steps called *Phase 1*. The algorithm is now split in two phases as follows.

Phase 1: (Determine a starting point x^0).

Step 1. If $\alpha_0(A + A^T) < 0$ then we set $F^0 := 0$. Otherwise, go to Step 3.

Step 2. Solve the following optimization problem with LMI constraints:

$$\begin{aligned} \min_{Q, X} \quad & \text{trace}(X) \\ \text{s.t.} \quad & A_{F^0} Q + Q A_{F^0}^T + B_1 B_1^T \prec 0, \\ & \begin{bmatrix} X & C_1 Q \\ QC_1^T & Q \end{bmatrix} \succ 0, \quad Q \succ 0, \end{aligned} \quad (27)$$

where $A_{F^0} := A + BF^0 C$. If this problem has a solution Q^0 and X^0 then terminate Phase 1 and using F^0 together with Q^0, X^0 as a starting point x^0 for Phase 2. Otherwise, go to Step 3.

Step 3. Solve the following feasibility problem with LMI constraints:

$$\begin{aligned} \text{Find } P \succ 0 \text{ and } K \text{ such that:} \\ \begin{bmatrix} PA + A^T P + KC + C^T K^T & PB_1 \\ B_1^T P & -\sigma_0^2 I_w \end{bmatrix} \preceq 0, \\ \begin{bmatrix} X & C_1 \\ C_1^T & P \end{bmatrix} \succeq 0, \end{aligned}$$

to obtain K^* and P^* , where σ_0 is a given regularization factor. Compute $F^* := B^+(P^*)^{-1}K^*$, where B^+ is a pseudo-inverse of B , and resolve problem (27) with $F^0 := F^*$. If problem (27) has a solution Q^0 and X^0 then set $x^0 := (F^0, Q^0, X^0)$ and terminate Phase 1. Otherwise, perform Step 4.

Step 4. Apply the method in Subsection V-B to solve the following BMI feasibility problem:

$$\begin{aligned} \text{Find } F \text{ and } Q \succ 0 \text{ such that :} \\ (A + BFC)Q + Q(A + BFC)^T + B_1 B_1^T \prec 0. \end{aligned} \quad (28)$$

If this problem has a solution F^0 then go back to Step 2. Otherwise, declare that no strictly feasible point is found.

Phase 2: (Solve problem (26)). Perform Algorithm 1 with the starting point x^0 found at Phase 1. \square

Note that Step 3 of Phase 1 corresponds to determining a full state feedback controller and approximating it subsequently with an output feedback controller. Step 4 of Phase 1 is usually time consuming. Therefore, in our numerical implementation, we terminate Step 4 after finding a point such that $\alpha_0((A + BFC)Q + Q(A + BFC)^T + B_1 B_1^T) \leq -0.1$.

Remark 5.2: The algorithm described in Phase 1 is finite. It is terminated either at Step 4 if no feasible point is found or at Step 2 if a feasible point is found. Indeed, if a feasible matrix F^0 is found at Step 4 then the first BMI constraint of (27) is feasible with some $Q \prec 0$. Thus we can find an appropriate matrix X such that $X - CQC^T \prec 0$, which implies the second LMI constraint of (27) is satisfied. Consequently, problem (27) has a solution.

The method used in Phase 1 is closely heuristic. It can be improved when we apply to a certain problem. However, as we can see in the numerical results, it performs quite acceptable for the majority of the test problems.

In the following numerical examples, we implement Phase 1 and Phase 2 of the algorithm using the decomposition

$$\begin{aligned} A_F Q + Q A_F^T + B_1 B_1^T &= \frac{1}{2} (A_F + Q)(A_F + Q)^T + B_1 B_1^T \\ &\quad - \frac{1}{2} (A_F - Q)(A_F - Q)^T \end{aligned}$$

for the BMI form at the left-top corner of the first constraint in (26). The regularization parameters and the stopping criterion for Algorithm 1 are chosen as in Subsection V-A and $K_{\max} := 300$. We test the algorithm for many problems in COMPl_{ib} and the computational results are reported in Table II. For the comparison purpose, we also carry out the test with HIFOO [12] and PENBMI [14], and the results are put in the columns marked by HIFOO and PENBMI in Table II, respectively. The initial controller for HIFOO is set to F^0 and the BMI parameters for PENBMI are initialized with $(F, Q, X) = (F^0, Q^0, X^0)$. Here, n, n_y, n_z, n_w, n_u are the dimensions of problems, the columns titled HIFOO and PENBMI give the \mathcal{H}_2 norm of the closed-loop system for the static output feedback controller computed by HIFOO and PENBMI; `iter` and `time[s]` are the number of iterations and CPU time in second of Algorithm 1, respectively, included Phase 1 and Phase 2. Problems marked by ‘‘b’’ mean that Step 4 in Phase 1 is performed. In Table II, we only report the

TABLE II
 \mathcal{H}_2 SYNTHESIS BENCHMARKS ON COMPL_EIB PLANTS

Name	Problem					Other Results, \mathcal{H}_2		Results and Performances		
	n_x	n_y	n_u	n_z	n_w	HIFOO	PENBMI	\mathcal{H}_2	iter	time[s]
AC1 ^b	5	3	3	2	3	0.0250	0.0061	0.0540	3	3.380
AC2 ^b	5	3	3	5	3	0.0257	0.0075	0.0540	3	3.390
AC3	5	4	2	5	5	2.0964	2.0823	2.1117	210	73.380
AC4	4	2	1	2	2	11.0269	-	11.0269	2	0.990
AC6	7	4	2	7	7	2.8648	2.8648	2.8664	153	124.230
AC7	9	2	1	1	4	0.0172	0.0162	0.0176	1	0.670
AC8	9	5	1	2	10	0.6330	0.7403	0.6395	300	282.760
AC12 ^b	4	4	3	1	3	0.0022	0.0106	0.0992	36	28.540
AC15 ^b	4	3	2	6	4	1.5458	1.4811	1.5181	264	85.390
AC16 ^b	4	4	2	6	4	1.4769	1.4016	1.4427	300	99.790
AC17	4	2	1	4	4	1.5364	1.5347	1.5507	171	49.350
HE2	4	2	2	4	4	3.4362	3.4362	4.7406	263	97.310
HE3 ^b	8	6	4	10	1	0.0197	0.0071	0.1596	249	217.360
HE4 ^b	8	6	4	12	8	6.6436	6.5785	7.1242	300	412.830
REA1	4	3	2	4	4	0.9442	0.9422	1.0622	249	80.810
REA2 ^b	4	2	2	4	4	1.0339	1.0229	1.1989	300	101.730
DIS1	8	4	4	8	1	0.6705	0.1174	0.7427	300	255.810
DIS2	3	2	2	3	3	0.4013	0.3700	0.3819	4	1.370
DIS3	6	4	4	6	6	0.9527	0.9434	1.0322	300	210.470
DIS4	6	6	4	6	6	1.0117	0.9696	1.0276	300	210.690
WEC1 ^b	10	4	3	10	10	7.3940	8.1032	12.9093	119	190.150
WEC2 ^b	10	4	3	10	10	6.7908	7.6502	12.2102	261	407.470
AGS	12	2	2	12	12	6.9737	6.9838	-	18	28.830
BDT1	11	3	3	6	1	0.0024	-	0.0017	51	64.410
MFP	4	2	3	4	4	6.9724	6.9724	7.0354	300	114.560
PSM	7	3	2	5	2	0.0330	0.0007	0.1753	300	217.250
EB2 ^b	10	1	1	2	2	0.0640	0.0084	0.1604	114	131.380
EB3	10	1	1	2	2	0.0732	0.0072	0.0079	7	6.240
TF1 ^b	7	4	2	4	1	0.0945	-	0.1599	192	166.810
TF2	7	3	2	4	1	11.1803	-	11.1803	3	2.810
TF3 ^b	7	3	2	4	1	0.1943	0.1424	0.2565	138	128.250
NN2	2	1	1	2	2	1.1892	1.1892	1.1892	4	1.090
NN4	4	3	2	4	4	1.8341	1.8335	1.8590	222	67.260
NN8	3	2	2	3	3	1.5152	1.5117	1.5725	183	50.170
NN11	16	5	3	3	3	0.1178	0.0790	0.1263	39	91.070
NN13 ^b	6	2	2	3	3	26.1012	26.1314	62.3995	138	112.750
NN14 ^b	6	2	2	3	3	26.1448	26.1314	62.3995	138	110.020
NN15	3	2	2	4	1	0.0245	-	0.0210	6	2.060
NN16 ^b	8	4	4	4	8	0.1195	0.1195	0.1195	3	23.030
NN17	3	1	2	2	1	3.2530	3.2404	3.3329	300	88.770

problems that were solved by Algorithm 1. The numerical results allow us to conclude that Algorithm 1, PENBMI and HIFOO report similar values for the majority of the test problems in COMPL_Eib.

If $D_{12} \neq 0$ then the third LMI constraint of (26) becomes a BMI constraint:

$$\begin{bmatrix} X & (C_1 + D_{12}FC)Q \\ Q(C_1 + D_{12}FC)^T & Q \end{bmatrix} \succeq 0, \quad (29)$$

which is equivalent to $X - C_F Q C_F^T \succeq 0$, where $C_F := C_1 + D_{12}FC$. Since $f(Q) := Q^{-1}$ is convex on $\mathbf{S}_{++}^{n_x}$ (see Lemma 3.1 a)), this BMI constraint can be reformulated as a convex-concave matrix inequality constraint of the form:

$$\begin{bmatrix} X & C_F \\ C_F^T & O \end{bmatrix} + \begin{bmatrix} O & O \\ O & Q^{-1} \end{bmatrix} \succeq 0. \quad (30)$$

By linearizing the concave term $-f(Q)$ at $Q = Q^k$ as $(Q^k)^{-1} - (Q^k)^{-1}(Q - Q^k)(Q^k)^{-1}$ (see [6]), the resulting constraint can be written as an LMI constraint. Therefore, Algorithm 1 can be applied to solve problem (29) in the case $D_{12} \neq 0$.

D. \mathcal{H}_∞ optimization: BMI formulation

Alternatively, we can also apply Algorithm 1 to solve the optimization with BMI constraints arising in \mathcal{H}_∞ optimization of the linear system (1). Let us assume that $D_{21} = 0$, then this problem is reformulated as the following optimization problem

with BMI constraints [17]:

$$\begin{aligned} \min_{F, X, \gamma} \quad & \gamma \\ \text{s.t.} \quad & \begin{bmatrix} A_F^T X + X A_F & X B_1 & C_F^T \\ B_1^T X & -\gamma I_w & D_{11}^T \\ C_F & D_{11} & -\gamma I_z \end{bmatrix} \prec 0, \\ & X \succ 0, \gamma > 0. \end{aligned} \quad (31)$$

Here, as before, $A_F = A + BFC$ and $C_F = C_1 + D_{12}FC$. The bilinear matrix term $A_F^T X + X A_F$ at the top-corner of the first constraint can be decomposed as (17) or (18). Therefore, we can use these decompositions to transform problem (31) into (9). After linearization, the resulting subproblem is also rewritten as a standard SDP problem by applying Lemma 3.2. We omit this specification here.

To determine a starting point, we perform Phase 1 which is similar to the one carried out in the \mathcal{H}_2 -optimization subsection.

Phase 1. (Determine a starting point $x^0 \in \text{ri}(\mathcal{D})$).

Step 1. If $\alpha_0(A^T + A) < 0$ then set $F^0 = 0$. Otherwise, go to Step 3.

Step 2. Solve the following optimization with LMI constraints

$$\begin{aligned} \min_{\gamma, X} \quad & \gamma \\ \text{s.t.} \quad & \begin{bmatrix} A_{F^0}^T X + X A_{F^0} & X B_1 & C_{F^0}^T \\ B_1^T X & -\gamma I_w & D_{11}^T \\ C_{F^0} & D_{11} & -\gamma I_z \end{bmatrix} \prec 0, \\ & X \succ 0, \gamma > 0, \end{aligned} \quad (32)$$

where $A_{F^0} := A + B F^0 C$ and $C_{F^0} := C_1 + D_{12} F^0 C$. If this problem has a solution γ^0 and X^0 then terminate Phase 1 and using F^0 together with γ^0, X^0 as a starting point x^0 for Phase 2. Otherwise, go to Step 3.

Step 3. Solve the following feasibility problem of LMI constraints:

$$\begin{aligned} \text{Find } P \succ 0, \gamma > 0 \text{ and } K \text{ such that:} \\ \begin{bmatrix} P A^T + A P + K^T B^T + B K & B_1 & P C_1 + K^T D_{12}^T \\ B_1^T & -\gamma I_w & D_{11}^T \\ C_1 P + D_{12} K & D_{11} & -\gamma I_z \end{bmatrix} \prec 0, \end{aligned}$$

to obtain K^*, γ^* and P^* . Compute $F^* := K^*(P^*)^{-1} C^+$, where C^+ is a pseudo-inverse of C , and resolve problem (32) with $F^0 := F^*$. If problem (32) has a solution X^0 and γ^0 then set $x^0 := (F^0, X^0, \gamma^0)$ and terminate Phase 1. Otherwise, perform Step 4.

Step 4. Apply the method in Subsection V-B to solve the following BMI feasibility problem:

$$\begin{aligned} \text{Find } F \text{ and } P \succ 0 \text{ such that:} \\ (A + BFC)^T P + P(A + BFC) \prec 0. \end{aligned} \quad (33)$$

If this problem has a solution F^0 then go back to Step 2. Otherwise, declare that no strictly feasible point for (31) is found. \square

As in the \mathcal{H}_2 problem, Phase 1 of the \mathcal{H}_∞ is also terminated after finite iterations. In this subsection, we also test this algorithm for several problems in COMPL_Eib using the same parameters and the stopping criterion as in the previous subsection. The computational results are shown in Table III. The

TABLE III
 \mathcal{H}_∞ SYNTHESIS BENCHMARKS ON COMPL_EIB PLANTS

Problem						Other Results, \mathcal{H}_∞		Results and Performances		
Name	n_x	n_y	n_u	n_z	n_w	HIFOO	PENBMI	\mathcal{H}_∞	iter	time[s]
AC1	5	3	3	2	3	0.0000	-	0.0177	93	28.050
AC2	5	3	3	5	3	0.1115	-	0.1140	99	32.540
AC3	5	4	2	5	5	4.7021	-	3.4859	210	76.170
AC4	4	2	1	2	2	0.9355	-	69.9900	2	2.620
AC6	7	4	2	7	7	4.1140	-	4.1954	167	138.570
AC7	9	2	1	1	4	0.0651	0.3810	0.0548	300	278.330
AC8	9	5	1	2	10	2.0050	-	3.0520	247	298.070
AC9 ^b	10	5	4	2	10	1.0048	-	0.9237	300	470.910
AC11 ^b	5	4	2	5	5	3.5603	-	3.0104	68	60.260
AC12	4	4	3	1	3	0.3160	-	2.3025	300	181.870
AC15	4	3	2	6	4	15.2074	427.4106	15.1995	105	36.700
AC16	4	4	2	6	4	15.4969	-	14.9881	186	68.820
AC17	4	2	1	4	4	6.6124	-	6.6373	129	42.400
HE1 ^b	4	1	2	2	2	0.1540	1.5258	0.1807	300	143.520
HE2	4	2	2	4	4	4.4931	-	6.7846	177	67.470
HE3	8	6	4	10	1	0.8545	1.6843	0.9243	105	95.000
HE4 ^b	8	6	4	12	8	23.3448	-	22.8713	252	325.580
HE5 ^b	8	2	4	4	3	8.8952	-	37.3906	300	430.820
REA1	4	3	2	4	4	0.8975	-	0.8815	96	34.430
REA2 ^b	4	2	2	4	4	1.1881	-	1.4188	300	118.320
REA3	12	3	1	12	12	74.2513	74.4460	74.5478	2	234.800
DIS1	8	4	4	8	1	4.1716	-	4.1943	93	66.130
DIS2	3	2	2	3	3	1.0548	1.7423	1.1546	54	17.120
DIS3	6	4	4	6	6	1.0816	-	1.1382	285	195.960
DIS4	6	6	4	6	6	0.7465	-	0.7498	126	93.220
TG1 ^b	10	2	2	10	10	12.8462	-	12.9336	45	84.380
AGS	12	2	2	12	12	8.1732	188.0315	8.1732	24	55.290
WEC2	10	4	3	10	10	4.2726	32.9935	6.6082	300	476.010
WEC3	10	4	3	10	10	4.4497	200.1467	6.8402	300	425.330
BDT1	11	3	3	6	1	0.2664	-	0.8562	29	40.910
MFP	4	2	3	4	4	31.5899	-	31.6079	171	57.430
IH	21	10	11	11	21	1.9797	-	1.1858	114	1206.340
CSE1	20	10	2	12	1	0.0201	-	0.0220	3	20.250
PSM	7	3	2	5	2	0.9202	-	0.9227	87	67.470
EB1	10	1	1	2	2	3.1225	39.9526	2.0276	300	295.420
EB2	10	1	1	2	2	2.0201	39.9547	0.8148	84	73.770
EB3	10	1	1	2	2	2.0575	3995311.0743	0.8153	84	75.820
NN1	3	2	1	3	3	13.9782	14.6882	18.4813	300	144.940
NN2	2	1	1	2	2	2.2216	-	2.2216	9	4.060
NN4	4	3	2	4	4	1.3627	-	1.3802	156	51.480
NN8	3	2	2	3	3	2.8871	78281181.1490	2.9345	180	51.830
NN9 ^b	5	2	3	4	2	28.9083	-	32.1222	300	129.920
NN11 ^b	16	5	3	3	3	0.1037	-	0.1566	9	366.890
NN15	3	2	2	4	1	0.1039	-	0.1194	6	3.740
NN16	8	4	4	4	8	0.9557	-	0.9656	48	37.950
NN17	3	1	2	2	1	11.2182	-	11.2381	117	32.160

numerical results computed by HIFOO and PENBMI are also included in Table III. Here, the notation is as same as in Table II, except that \mathcal{H}_∞ denotes the \mathcal{H}_∞ -norm of the closed-loop system for the static output feedback controller. We can see from Table III that the optimal values reported by Algorithm 1 and HIFOO are almost similar for many problems whereas in general PENBMI has difficulties in finding a feasible solution.

E. $\mathcal{H}_2/\mathcal{H}_\infty$ optimization: BMI formulation

Motivated from the \mathcal{H}_2 and \mathcal{H}_∞ optimization problems, in this subsection we consider the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis problem. Let us assume that $D_{11} = 0$, $D_{21} = 0$ and the performance output z is divided in two components, z_1 and z_2 . Then the linear system (1) becomes:

$$\begin{cases} \dot{x} = Ax + B_1 w + Bu, \\ z_1 = C_1^{z_1} x + D_{12}^{z_1} u, \\ z_2 = C_1^{z_2} x + D_{12}^{z_2} u, \\ y = Cx. \end{cases} \quad (34)$$

The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem is to find a static output feedback gain F such that, for $u = Fy$, the \mathcal{H}_2 -norm of the closed loop from w to z_2 is minimized, while the \mathcal{H}_∞ -norm from w to z_1 is less than some imposed level γ [5], [18], [23].

This problem leads to the following optimization problem

with BMI constraints [23]:

$$\begin{aligned} \min_{F, P_1, P_2, Z} \quad & \text{trace}(Z) \\ \text{s.t.} \quad & \begin{cases} A_F^T P_1 + P_1 A_F + (C_F^{z_1})^T C_F^{z_1} P_1 B_1 - \gamma^2 I \\ B_1^T P_1 \\ A_F^T P_2 + P_2 A_F P_2 B_1 \\ B_1^T P_2 \\ P_2 \quad (C_F^{z_2})^T \\ C_F^{z_2} \quad Z \end{cases} \prec 0, \\ & P_1 \succ 0, P_2 \succ 0, \end{cases} \quad (35) \end{aligned}$$

where $A_F := A + BFC$, $C_F^{z_1} := C_1^{z_1} + D_{12}^{z_1} FC$ and $C_F^{z_2} := C_1^{z_2} + D_{12}^{z_2} FC$. Note that if $C = I_{n_x}$, the identity matrix, then this problem becomes a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ of static state feedback design problem considered in [23]. In this subsection, we test Algorithm 1 for the static state feedback and output feedback cases.

Case 1. The static state feedback case ($C = I_{n_x}$). First, we apply the method in [23] to find an initial point via solving two optimization problems with LMI constraints. Then, we use the same approach as in the previous subsections to transform problem (35) into an optimization problem with psd-convex-concave matrix inequality constraints. Finally, Algorithm 1 is implemented to solve the resulting problem. For convenience of implementation, we introduce a slack variable η and then replace the objective function in (31) by $f(x) = \eta^2$ with an additional constraint $\text{trace}(Z) \leq \eta^2$.

In the first case, we test Algorithm 1 with three problems. The first problem was also considered in [13] with

$$A = \begin{bmatrix} -1.40 & -0.49 & -1.93 \\ -1.73 & -1.69 & -1.25 \\ 0.99 & 2.08 & -2.49 \end{bmatrix}, B_1 = \begin{bmatrix} -0.16 & -1.29 \\ 0.81 & 0.96 \\ 0.41 & 0.65 \end{bmatrix}, B = \begin{bmatrix} 0.25 \\ 0.41 \\ 0.65 \end{bmatrix}, \\ C_1^{z_1} = [-0.41 \quad 0.44 \quad 0.68], C_1^{z_2} = [-1.77 \quad 0.50 \quad -0.40], \\ D_{12}^{z_1} = D_{12}^{z_2} = 1, \text{ and } \gamma = 2.$$

If the tolerance $\varepsilon = 10^{-3}$ is chosen then Algorithm 1 converges after 17 iterations and reports the value $\eta = 0.7489$ with $F = [1.9485 \quad 0.3990 \quad -0.2119]$. This result is similar to the one shown in [23]. If we regularize the subproblem (12) with $\rho = 0.5 \times 10^{-3}$ and $Q = I_{PF}$ then the number of iterations is reduced to 10 iterations.

The second problem is DIS4 in COMPL_EIB [17]. In this problem, we set $C_1^{z_1} = C_1^{z_2}$ and $D_{12}^{z_1} = D_{12}^{z_2}$ as in [23]. Algorithm 1 converges after 24 iterations with the same tolerance $\varepsilon = 10^{-3}$. It reports $\eta = 1.6925$ and $\gamma = 1.1996$ with

$$F = \begin{bmatrix} -0.8663 & -0.6504 & -1.1115 & -0.1951 & -0.6099 & 0.2065 \\ 0.1591 & -0.4941 & -0.6322 & -0.5409 & -1.2895 & 0.2774 \\ -0.7017 & -0.0785 & 0.6121 & -0.8919 & 0.2518 & -0.2354 \\ -0.0522 & -0.5556 & -0.5838 & 0.4497 & -1.4279 & -0.6677 \end{bmatrix}.$$

If we regularize the subproblem (12) with $\rho = 0.5 \times 10^{-3}$ and $Q = I_{PF}$ then the number of iterations is 18.

The third problem is AC16 in COMPL_EIB [17]. In this example we also choose $C_1^{z_1} = C_1^{z_2}$ and $D_{12}^{z_1} = D_{12}^{z_2}$ as in the previous problem. As mentioned in [23], if we choose a starting value $\gamma_0 = 100$, then the LMI problem can not be solved by the SDP solvers (e.g., Sedumi, SDPT3) due to numerical problems. Thus, we rescale the LMI constraints using the same trick as in [23]. After doing this, Algorithm 1 converges after 298 iterations with the same tolerance

$\varepsilon = 10^{-3}$. The value of η reported in this case is $\eta = 12.3131$ and $\gamma = 20.1433$ with

$$F = \begin{bmatrix} -1.8533 & 0.1737 & 0.6980 & 6.4208 \\ 4.2672 & -0.9668 & -1.5952 & -2.9240 \end{bmatrix}.$$

The results obtained by Algorithm 1 for solving problems DIS4 and AC16 in this paper confirm the results reported in [23].

Case 2. The static output feedback case. As before, we first propose a technique to determine a starting point for Algorithm 1. We described this phase algorithmically as follows.

Phase 1. (Determine a starting point x^0).

Step 1. If $\alpha_0(A^T + A) < 0$ then set $F^0 = 0$. Otherwise, go to Step 3.

Step 2. Solve the following linear SDP problem:

$$\begin{aligned} & \min_{P_1, P_2, Z} \text{trace}(Z) \\ & \text{s.t.} \begin{cases} \begin{bmatrix} A_{F^0}^T P_1 + P_1 A_{F^0} + (C_{F^0}^{z_1})^T C_{F^0}^{z_1} & P_1 B_1 \\ B_1^T P_1 & -\gamma^2 I \end{bmatrix} \prec 0, \\ \begin{bmatrix} A_{F^0}^T P_2 + P_2 A_{F^0} & P_2 B_1 \\ B_1^T P_2 & -I \end{bmatrix} \prec 0, \\ \begin{bmatrix} P_2 & (C_{F^0}^{z_2})^T \\ C_{F^0}^{z_2} & Z \end{bmatrix} \succ 0, \quad P_1 \succ 0, \quad P_2 \succ 0, \end{cases} \end{aligned} \quad (36)$$

where $A_{F^0} = A + BF^0C$, $C_{F^0}^{z_1} = C_1^{z_1} + D_{12}^{z_1}F^0C$ and $C_{F^0}^{z_2} = C_1^{z_2} + D_{12}^{z_2}F^0C$. If this problem has an optimal solution P_1^0, P_2^0 and Z^0 then terminate Phase 1. Set $x^0 := (F^0, P_1^0, P_2^0, Z^0)$ for a starting point of Algorithm 1 in Phase 2. Otherwise go to Step 3.

Step 3. Solve the following LMI feasibility problem:

$$\begin{aligned} & \text{Find } Q \succ 0, W \text{ and } Z \text{ such that:} \\ & \begin{cases} \begin{bmatrix} AQ + QA^T + BW + W^T B^T & B_1 & (C_1 + D_{12}W)^T \\ B_1^T & -I_w & O \\ C_1 + D_{12}W & O & -\gamma^2 I_z \end{bmatrix} \prec 0, \\ \begin{bmatrix} AQ + QA^T + BW + W^T B^T & B_1 \\ B_1^T & -I_w \end{bmatrix} \prec 0, \\ \begin{bmatrix} Q & (C_1Q + D_{12}W)^T \\ C_1Q + D_{12}W & Z \end{bmatrix} \succ 0, \end{cases} \end{aligned} \quad (37)$$

to obtain a solution Q^* , W^* and Z^* . Set $F^* := W^*(Q^*)^{-1}C^+$, where C^+ is the pseudo-inverse of C . Solve again problem (36) with $F^0 := F^*$. If problem (36) has solution then terminate Phase 1. Otherwise, perform Step 4.

Step 4. Solve the following optimization with BMI constraints:

$$\begin{aligned} & \max_{\beta, F, P_1, P_2} \beta \\ & \text{s.t. } P_1 \succ 0, \quad P_2 \succ 0, \\ & A_F^T P_1 + P_1 A_F + (C_F^{z_1})^T C_F^{z_1} + \frac{1}{\gamma^2} P_1 B_1 B_1^T P_1 \preceq -2\beta P_1, \\ & A_F^T P_2 + P_2 A_F + P_2 B_1 B_1^T P_2 \preceq -2\beta P_2 \end{aligned} \quad (38)$$

to obtain an optimal solution F^* corresponding to the optimal value β^* . If $\beta^* > 0$ then set $F^0 := F^*$ and go back to Step 2 to determine P_1^0, P_2^0 and Z^0 . Otherwise, declare that no strictly feasible point of problem (35) is found. \square

Since at Step 4 of Phase 1, it requires to solve an optimization problem with two BMI constraints. This task is usually expensive. In our implementation, we only terminate

this step after find a strictly feasible point with a feasible gap 0.1. If matrix C is invertible then matrix F^* at Step 3 is $F^* = W^*(Q^*)^{-1}C^{-1}$. Hence, we can ignore Step 4 of Phase 1.

To avoid the numerical problem in Step 3, we can reformulate problem (37) equivalently to the following one:

Find $Q \succ 0, W$ and Z such that:

$$\begin{cases} \begin{bmatrix} AQ + QA^T + BW + W^T B^T & B_1 & (C_1 + D_{12}W)^T \\ B_1^T & -\gamma I_w & O \\ C_1 + D_{12}W & O & -\gamma I_z \end{bmatrix} \prec 0, \\ \begin{bmatrix} Q & (C_1Q + D_{12}W)^T \\ C_1Q + D_{12}W & Z \end{bmatrix} \succ 0. \end{cases}$$

We test the algorithm described above for several problems in COMPL_EIB with the level values $\gamma = 4$ and $\gamma = 10$. In these examples, we assume that the output signals $z_1 \equiv z_2$. Thus we have $C_1^{z_1} = C_1^{z_2} = C_1$ and $D_{12}^{z_1} = D_{12}^{z_2} = D_{12}$. The parameters and the stopping criterion of the algorithm are chosen as in Subsection V-C. The computational results are reported in Table IV with $\gamma = 4$ and $\gamma = 10$. Here,

TABLE IV
 $\mathcal{H}_2/\mathcal{H}_\infty$ SYNTHESIS BENCHMARKS ON COMPL_EIB PLANTS

Problem	Results and Performances ($\gamma = 4$)			Results and Performances ($\gamma = 10$)			
	Name	$\mathcal{H}_2/\mathcal{H}_\infty$	iter	time[s]	$\mathcal{H}_2/\mathcal{H}_\infty$	iter	time[s]
AC1	0.0585 / 0.0990	3	4.22	0.0585 / 0.0990	3	4.27	
AC2	0.1067 / 0.1723	6	7.31	0.1070 / 0.1727	3	7.15	
AC3	5.2770 / 3.9999	51	281.53	4.5713 / 5.1298	18	19.18	
AC6	- / -	-	-	4.0297 / 4.8753	283	330.64	
AC7	0.0415 / 0.0961	1	3.39	0.0420 / 0.1286	2	3.91	
AC8	1.2784 / 2.2288	43	60.78	1.3020 / 2.5719	23	31.59	
AC11	4.0704 / 4.0000	76	175.75	4.0021 / 5.1949	117	122.86	
AC12	0.0924 / 0.3486	18	73.46	1.4454 / 1.6444	300	234.13	
AC17	- / -	-	-	4.1228 / 6.6472	2	11.620	
HE1	0.1123 / 0.2257	2	131.18	0.0973 / 0.2080	1	30.97	
HE2	- / -	-	-	4.7302 / 9.8931	75	55.48	
REA1	1.8214 / 1.4740	30	25.64	1.8213 / 1.4730	30	26.65	
REA2	3.5014 / 3.5180	42	22.09	3.5015 / 3.5209	45	23.26	
DIS1	- / -	-	-	2.8505 / 4.7904	15	30.51	
DIS2	1.5079 / 1.8500	18	7.92	1.5079 / 1.8520	21	7.92	
DIS3	2.0577 / 1.7934	27	25.03	2.0577 / 1.7766	30	24.54	
DIS4	1.6926 / 1.1952	21	18.62	1.6926 / 1.2009	24	21.55	
AGS	- / -	-	-	7.0332 / 8.2035	8	196.73	
PSM	1.5115 / 0.9248	177	160.41	1.5115 / 0.9248	180	167.31	
EB2	0.7765 / 1.0828	7	9.70	0.7768 / 1.0867	10	13.16	
EB3	0.8406 / 0.9249	1	3.21	0.8383 / 0.9418	1	2.93	
EB4	1.0147 / 1.0707	6	59.55	0.9981 / 1.2146	12	111.26	
NN2	1.5651 / 2.4834	12	5.37	1.5651 / 2.4876	12	5.49	
NN4	1.8778 / 2.0501	202	154.49	1.8779 / 2.0519	213	161.00	
NN8	2.3609 / 3.9999	21	15.71	2.3376 / 4.6514	15	6.57	
NN15	0.0820 / 0.1010	42	18.75	0.0771 / 0.1012	24	10.47	
NN16	0.3187 / 0.9574	90	96.44	0.3319 / 0.9572	258	303.87	

$\mathcal{H}_2/\mathcal{H}_\infty$ are the \mathcal{H}_2 and \mathcal{H}_∞ norms of the closed-loop systems for the static output feedback controller, respectively. With $\gamma = 10$, the computational results show that Algorithm 1 satisfies the condition $\|P_\infty(s)\|_\infty \leq \gamma = 10$ for all the test problems. While, with $\gamma = 4$, there are 5 problems reported infeasible, which are denoted by "-". The \mathcal{H}_∞ -constraint of three problems AC3, AC11 and NN8 is active with respect to $\gamma = 4$.

VI. CONCLUDING REMARKS

We have proposed a new algorithm for solving many classes of optimization problems involving BMI constraints arising in static feedback controller design. The convergence of the algorithm has been proved under standard assumptions. Then, we have applied our method to design static feedback controllers for various problems in robust controller design. The algorithm is easy to implement using the current SDP

software tools. The numerical results are also reported for the benchmark collection in COMPl_eib. The algorithm requires a strictly feasible starting point which is determined by Phase 1. This phase is implemented based on some heuristic techniques which may need to solve a feasibility problem with BMI constraints. In the previous numerical examples, Phase 1 costs from 0.1% up to 41% of the total time depending on each problem.

Note, however, that our method depends crucially on the psd-convex-concave decomposition of the BMI constraints. In practice, it is important to look at the specific structure of the problems and find an appropriate psd-convex-concave decomposition for Algorithm 1. The method proposed can be extended to general nonlinear semidefinite programming, where the psd-convex-concave decomposition of the nonconvex mappings are available. From a control design point of view, the application to more general reduced order controller synthesis problems and the extension towards linear parameter varying or time-varying systems are future research directions.

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A. Proof of Lemma 4.2.

For any matrices $A, B \in \mathcal{S}_+^p$, we have $A \circ B \geq 0$. From Step 1 of Algorithm 1, we have x^{k+1} is a solution of the convex subproblem (12) and Λ^{k+1} is the corresponding multiplier, under Assumption 3, they must satisfy the following generalized Kuhn-Tucker condition:

$$\begin{cases} 0 \in \nabla f(x^{k+1}) + \rho_k Q_k^T Q_k (x^{k+1} - x^k) + \left\{ \sum_{i=1}^l D[G_i(x) - H_i(x^k) - DH_i(x^k)(x - x^k)] \right\} \Lambda_i^{k+1} + \mathcal{N}_\Omega(x^{k+1}), \\ G_i(x^{k+1}) - H_i(x^k) - DH_i(x^k)(x^{k+1} - x^k) \leq 0, \Lambda_i \geq 0, \\ [G_i(x^{k+1}) - H_i(x^k) - DH_i(x^k)(x^{k+1} - x^k)] \circ \Lambda_i^{k+1} = 0. \end{cases} \quad (39)$$

Noting that $D[G_i(x) - H_i(x^k) - DH_i(x^k)(y - x^k)]|_{x=x^{k+1}} = DG_i(x^{k+1}) - DH_i(x^k)$ for $i = 1, \dots, l$, it follows from the first line of (39) and the convexity of f that

$$\begin{aligned} f(y) - f(x^{k+1}) &+ \left\{ \sum_{i=1}^l [DG_i(x^{k+1}) - DH_i(x^k)] \Lambda_i^{k+1} \right\}^T (y - x^{k+1}) \\ &\geq \left\{ \nabla f(x^{k+1}) + \sum_{i=1}^l [DG_i(x^{k+1}) - DH_i(x^k)] \Lambda_i^{k+1} \right\}^T (y - x^{k+1}) \\ &+ \frac{\rho_f}{2} \|y - x^{k+1}\|_2^2 \\ &\geq \frac{\rho_f}{2} \|y - x^{k+1}\|_2^2 + \rho_k (y - x^{k+1})^T Q_k^T Q_k (x^k - x^{k+1}), \quad \forall y \in \Omega. \end{aligned} \quad (40)$$

On the other hand, we have

$$\begin{aligned} &\left\{ [DG_i(x^{k+1}) - DH_i(x^k)] \Lambda_i^{k+1} \right\}^T (y - x^{k+1}) \\ &= \Lambda_i^{k+1} \circ [DG_i(x^{k+1})(y - x^{k+1}) - DH_i(x^k)(y - x^{k+1})]. \end{aligned} \quad (41)$$

Since G_i and H_i are psd-convex, applying Lemma 2.2 we have

$$\begin{aligned} G_i(x^k) - G_i(x^{k+1}) &\succeq DG_i(x^{k+1})(x^k - x^{k+1}), \\ \text{and } H_i(x^{k+1}) - H_i(x^k) &\succeq DH_i(x^k)(x^{k+1} - x^k), \quad i = 1, \dots, l. \end{aligned}$$

Summing up these inequalities we obtain

$$\begin{aligned} G_i(x^k) - H_i(x^k) - [G_i(x^{k+1}) - H_i(x^{k+1})] \\ \succeq [DG_i(x^{k+1})(x^k - x^{k+1}) - DH_i(x^k)(x^k - x^{k+1})]. \end{aligned}$$

Using the fact that $\Lambda_i^{k+1} \succeq 0$, this inequality implies that

$$\begin{aligned} \Lambda_i^{k+1} \circ \{G_i(x^k) - H_i(x^k) - [G_i(x^{k+1}) - H_i(x^{k+1})]\} \\ \succeq \Lambda_i^{k+1} \circ [DG_i(x^{k+1})(x^k - x^{k+1}) - DH_i(x^k)(x^k - x^{k+1})]. \end{aligned} \quad (42)$$

Substituting $y = x^k$ into (40) and then combining the consequence, (41), (42) and the last line of (39) to obtain

$$\begin{aligned} f(x^k) - f(x^{k+1}) &+ \sum_{i=1}^l \Lambda_i^{k+1} \circ [G_i(x^k) - H_i(x^k)] \\ &\geq \frac{\rho_f}{2} \|x^{k+1} - x^k\|_2^2 + \rho_k \|Q_k(x^{k+1} - x^k)\|_2^2. \end{aligned} \quad (43)$$

Now, since x^k is the solution of the convex subproblem (12) linearized at x^{k-1} . One has $G_i(x^k) - H_i(x^k) \leq 0$. Moreover, since $\Lambda_i^{k+1} \succeq 0$, we have $\Lambda_i^{k+1} \circ [G_i(x^k) - H_i(x^k)] \leq 0$. Substituting this inequality into (43), we obtain

$$f(x^k) - f(x^{k+1}) \geq \frac{\rho_f}{2} \|x^k - x^{k+1}\|_2^2 + \rho_k \|Q_k(x^{k+1} - x^k)\|_2^2.$$

This inequality is indeed (14) which proves the item a). If there exists at least one $i_0 \in \{1, \dots, l\}$ such that $G_{i_0}(x^k) - H_{i_0}(x^k) < 0$ and $\Lambda_{i_0}^{k+1} \succ 0$ then $\Lambda_{i_0}^{k+1} \circ [G_{i_0}(x^k) - H_{i_0}(x^k)] < 0$. Substituting this inequality into (43) we conclude that $f(x^{k+1}) < f(x^k)$ which proves item b). The last statement c) follows directly from the inequality (14). \square

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